CS 3313 Foundations of Computing:

Properties of Regular Languages: Non-regular Languages

http://gw-cs3313-2021.github.io

Properties of Regular Languages

- Closure Properties: what happens when we “combine” two regular languages or perform set operations on them ?
  - Ex: Is Intersection of two regular languages still a regular language ?
  - Why is this important ?
    - Construct a larger set from smaller sets
    - Problem decomposition
- Decision Problems: can we provide procedures to determine properties of a language ?
  - Ex: are two machines equivalent? Does a DFA accept an infinite set ?
- How do we determine if a language does not belong to that class of languages ?
  - Ex: How do we show that a language (problem?) cannot be accepted by a DFA ?

Frequently used concept: Product DFA

- "compose" two DFAs using cartesian product of their states
- Let M_1 and M_2 be two DFAs with states Q and R
  - M_1= (Q, Σ, $\delta_{1,}$ q_0, F_1) and M_1= (R, Σ, $\delta_{2,}$ r_0, F_2)
- Product DFA M_p:
- Product DFA has set of states $Q \times R$
  - i.e., pairs *[q,r]* with q in Q and r in R
- Start state = [q_0, r_0] (the start states of the two DFA's).
- Transitions: $\delta([q,r], a) = [\delta_1(q,a), \delta_2(r,a)]$
  - δ_1, δ_2 are the transition functions for the DFA's of M_1, M_2
  - That is, ***we simulate the two DFA's in the two state components of the product DFA.***
- Note: we have not yet defined the final states of the product DFA

Summary of Closure Properties

- Regular languages are closed under Union, Concatenation, star closure, complementation, reversal, intersection, homomorphism (and reverse homomorphisms)

- Where are closure properties used ?
  - Construction a solution (DFA or Reg. Expr.) for a larger language using simpler solutions (machines or languages)
    - Analogy: modular composition of software modules
  - Useful in simplifying proofs to show a language is not regular
  - Useful in constructing "decision algorithms"

Decision Properties of Regular Languages

- A *decision property* for a class of languages is an algorithm that takes a formal description of a language (e.g., a DFA) and tells whether or not some property holds.
- We say that the property is *decidable* if there is an algorithm that determines if the property holds

Decision Properties for Regular Languages

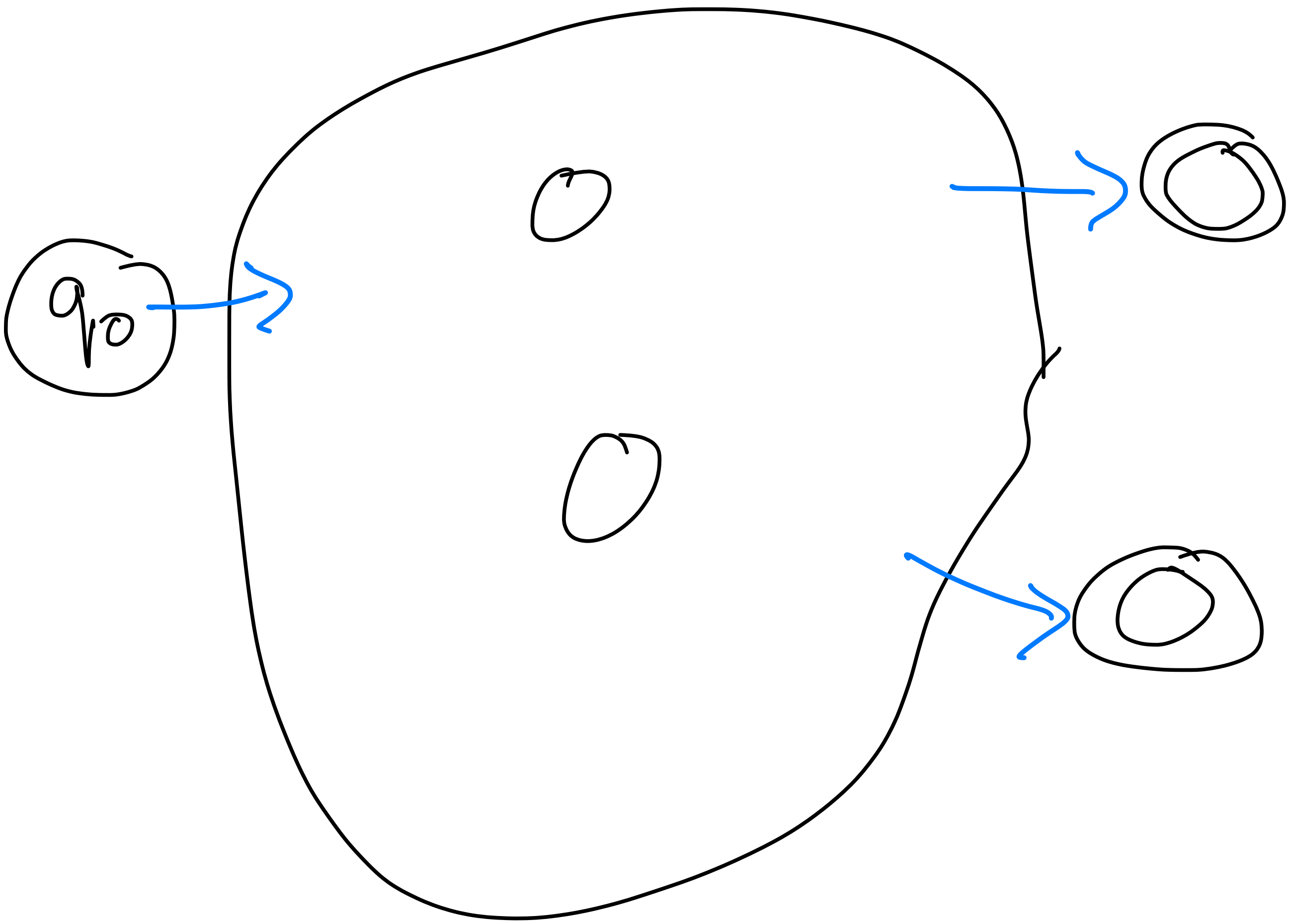
- Membership: Is w in L(M) ?
- Emptiness: Is L(M) empty ?
- Equivalence: Is L(M1) = L(M2) ?
- Subset: Is L(M1) a subset of L(M2) ?
- Infiniteness: Is L(M) infinite ?

The Infiniteness Problem

- Is a given regular language infinite?
- Theorem: Testing if L(M) is infinite is a decidable problem.
- Start with a DFA for the language.
- Key idea: if the DFA has *n* states, and the language contains any string of length *n* or more, then the language is infinite.
- Otherwise, the language is surely finite.
  - Limited to strings of length *n* or less.

L(M) infinite ?

- Proof: use the graph representation to present the procedure/proof.

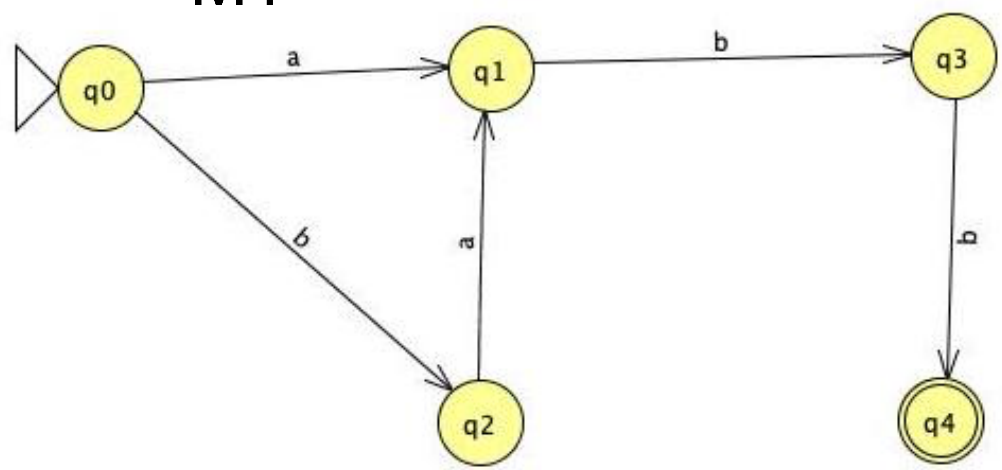


Transition graphs for two DFAs

Is L(M1) finite ?

Yes

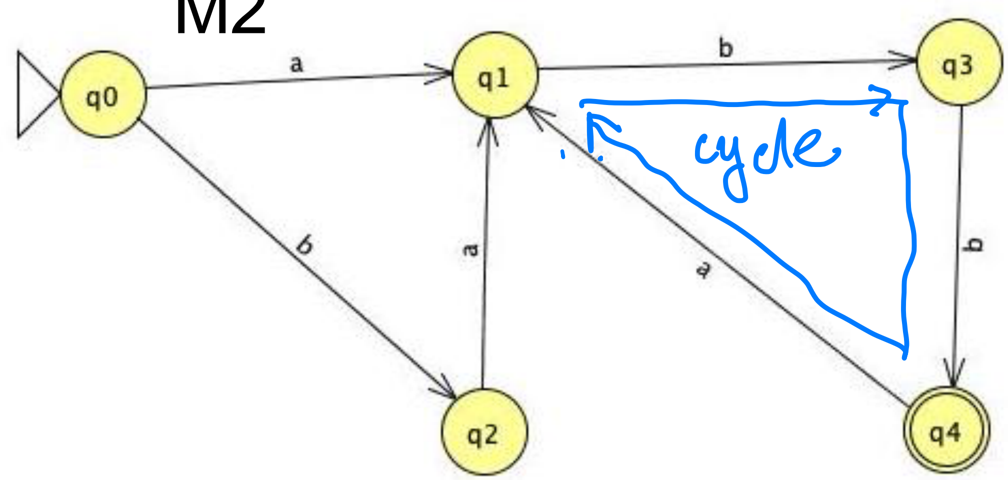
M1



Is L(M2) finite ?

No –
cycle in the graph

M2



Algorithm to test for L(M) infinite

- Input: Transition graph for DFA M
- Output: Yes if L(M) is infinite, No if L(M) is finite
- Algorithm ?
- Check if graph has a cycle!

So what kinds of languages are not regular and how do we prove they are not ?

- Proof for testing infiniteness of L(M) reveals some properties that can be used to prove that a language is not regular.

- Given any language L, it is either regular or it is not.
  - To prove L is regular, we have to provide a DFA/NFA or Regular expression that accepts L.
  - To prove L is not regular, we need to provide a formal proof using some properties of all regular languages
    - Simply saying “I spent a lot of time and could not find a DFA” is NOT a proof.

Why is it useful to ask if a language is Regular (or Context free or …)

- Example: Can we check if there are syntax errors in a C program by using a DFA ?
  - Syntax checking is first step in a compiler's translation process
  - Program must satisfy the rules (specified as a grammar) of the C programming language (or any programming language)

Power of abstraction

- If a DFA can do syntax checking, then a DFA can check if there are an equal number of left and right braces ({ and } are used to specify a code block in C)
  - Choose L = { w | w is a string over a,b and w has equal number of a's and b's}
    - Using a to denote { and b to denote } (recall homomorphism which will let you substitute symbols)

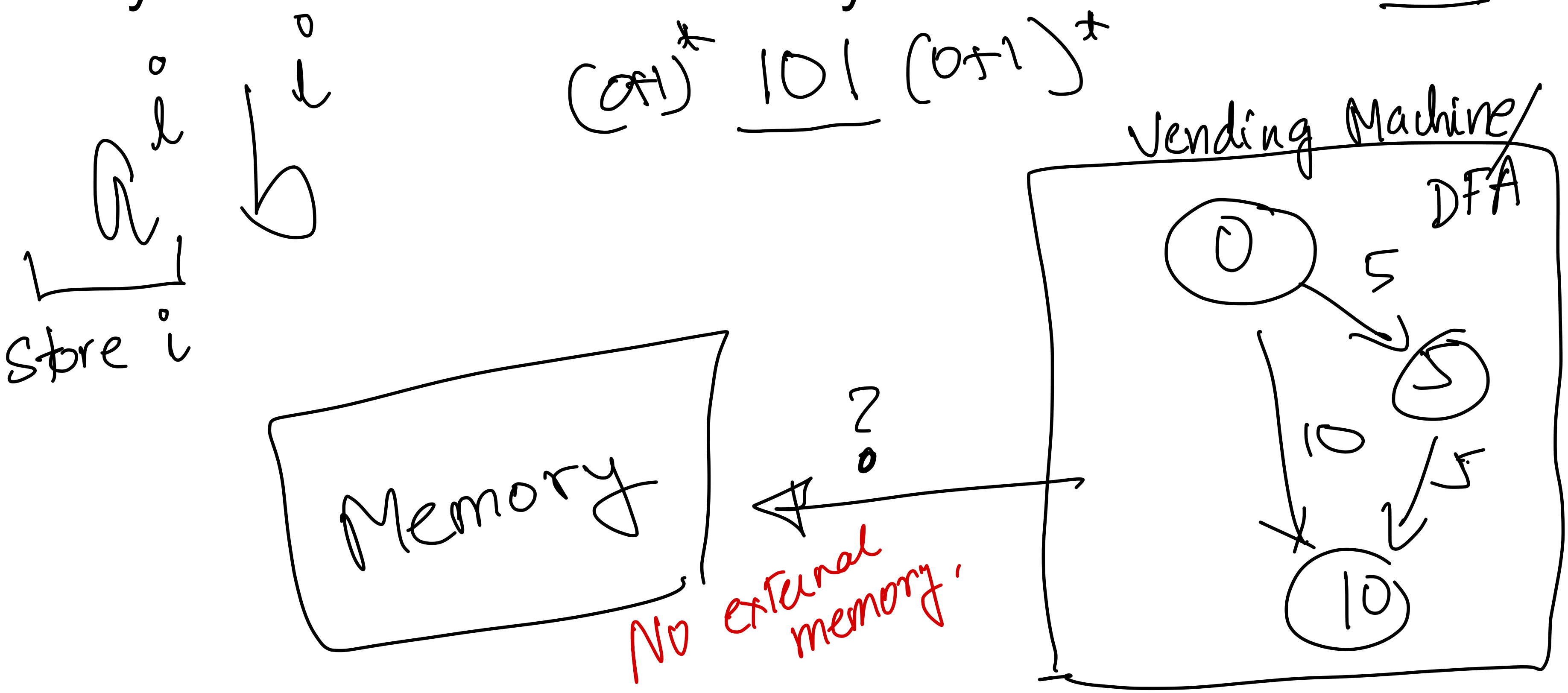
- Now apply closure properties: we know that a*b* is regular and regular languages are closed under intersection
  - Therefore **L1 = L ∩ a*b* = {a^i b^i | i >1}** must be a regular language
    - Equal number of a's and b's
- So, is L1 a regular language ?

"power" of DFAs: A little intuition

- So what can DFAs (i.e., finite state machines) "compute" ?
- What can they store and where ?
  - State

State: Summarizes what has occured thus far.

- Do they have an "external" memory to store a value ? NO

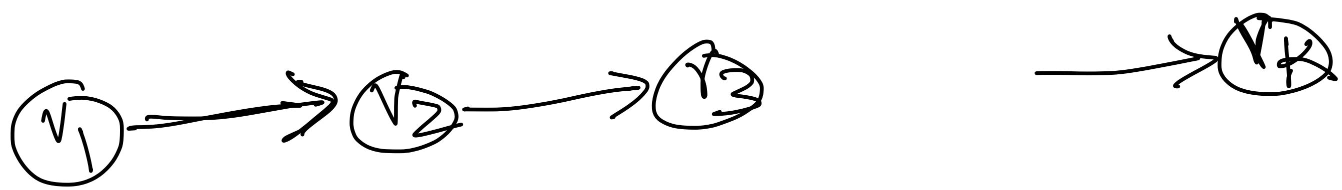


Paths in a graph

$G = (V, E)$

$|V| = n$

- Graph has n vertices
- Path in a graph can be represented as a sequence of vertices: $v_1 v_2 v_3 \ldots v_k$ where (v_i, v_j) is an edge
- Suppose we have a path of length n, how many vertices on the path ?



$v_1 v_2 \ldots v_n v_{n+1}$ $\Rightarrow$ $(n+1)$ vertices on the path

path of length n.

DFA Transition Graph

- The transition function of a DFA can be represented as a (directed graph).
- DFA has a finite number of states: n
- Suppose there is a string of length >= n accepted by the DFA
  - Vertex sequence in the path = ? (n+1) vertices visited on the path

i.e., $p_1 \; p_2 \; p_3 \; \ldots \; p_n \; p_{n+1}$, and $p_i \in Q$

$\delta(q_0, w) \in F$

$p_1 = q_0$ and $p_{n+1} \in F$

Cycles in the path

$M = (Q, \Sigma, \delta, q_0, F)$ $|Q| = n$

Vertex sequence in path = $p_1 p_2 \cdots p_i \cdots p_j \cdots p_n p_{n+1}$

$p_1 = q_0$ and each $p_i \in Q$ and $p_{n+1} \in F$

$\therefore$ from pigeon hole principle, we have n unique states|vertices

$\therefore$ 2 states from $(p_1, p_2 \cdots p_{n+1})$ are the same

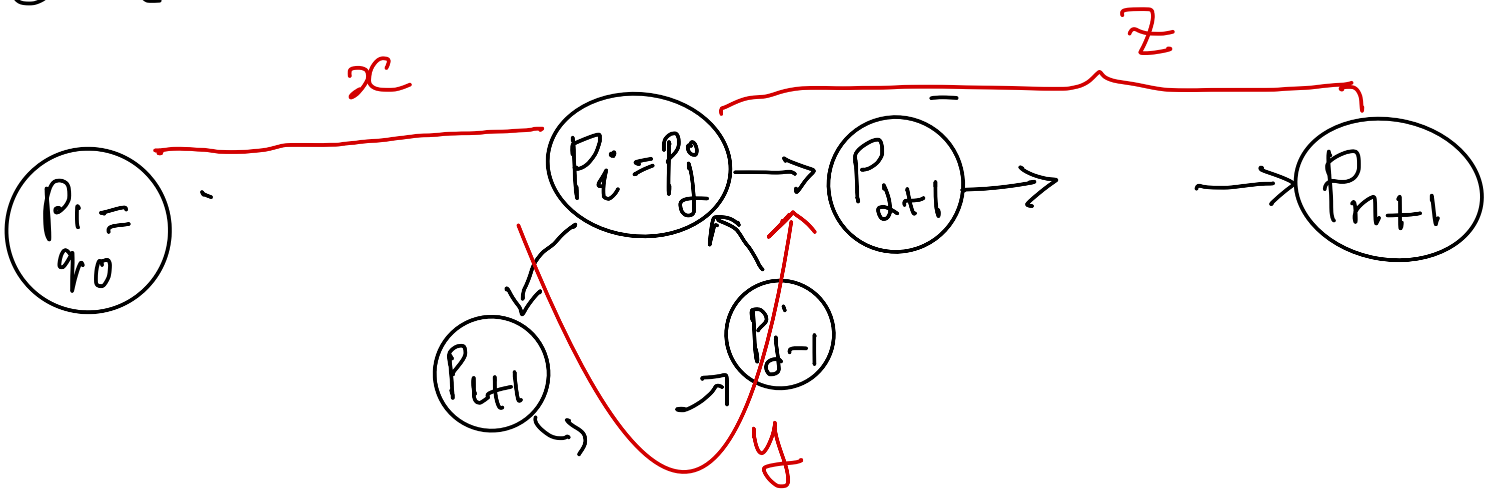
there must i, j such that $j > i$ $1 \le i, j \le n+1$

and $p_i = p_j$

the path in DFA. for input w is

$p_1 p_2 \cdots p_i p_{i+1} \cdots p_{j-1} p_i p_{j+1} \cdots p_{n+1}$

$w \in L(M)$



$w = xyz$, $|y| \geq 1$

$|xy| \leq n$

$\delta(p_i, y) = p_i$

$\delta(q_0, x) = p_i$

$\delta(p_i, z) \in F$

$\therefore \delta(q_0, xy^i z) \in F$ for all $i \geq 0$.

$\therefore$ if $w \in L$ then $xy^i z \in L$.

The Pumping Lemma for Regular Languages

For every regular language L

There is an integer n, such that

For every string w in L of length $\geq$ n

We can write w = xyz such that:

1. $|xy| \leq n$.
2. $|y| > 0$.
3. For all $i \geq 0$, xy^iz is in L.

Number of states of DFA for L

Labels along first cycle on path labeled w

Example: L= { $a^i b^i$ | i>= 0 }

Assume L is regular, then $\exists$ n, constant.

let $w = a^n b^n$

$w = xyz$, $|xy| \leq n$ and $|y| \geq 1$

$\therefore$ xy consists entirely of a's, x has length m_1

y has length m_2

$m_2 \geq 1$

From lemma

$$xy^0z \in L$$

$$xy^0z = a^{m_1} . \lambda . a^{n-m_1-m_2} b^n$$

$$= a^{n-m_2} b^n, \quad \text{but } m_2 \geq 1$$

$\therefore$ $n - m_2 \neq n \Rightarrow xy^0z \in L$

- contradiction.

How do use the pumping lemma: 2 person adversarial game

- **For all** regular languages L, **there exists** *n*…**for all** *w* in L

….**there exists** *xyz*….

- Logical statements/assertions that have several alternations of for all and there exists quantifiers can be thought of as a game between two players

- Application of the pumping lemma can be seen as a two player game (of 5 steps)
- Example: L = { ww | w in {a,b}* }

Pumping Lemma as Adversarial Game

- 1: Player 1 (me) picks the language to be proved nonregular

  $L = \{ww \mid w \in \{a,b\}^*\}$

- 2. Player 2 picks *n*, but does'nt reveal to player 1 what n is; player 1 must devise a play for all possible *n's*

- 3. Player 1 picks *w*, which may depend on n and which must be of length at least *n*

  $w = a^n b^n a^n b^n$

Pumping Lemma as Adversarial Game

- 4: Player 2 divides w into x,y,z obeying the constraints that are stipulated in the lemma: y is not empty and |xy| <= n.
  - Again, Player 2 does not tell Player 1 what xyz are; just that they obey the constraints

$w = xyz$

$y \neq \lambda$ and $|xy| \leq n$

Let $|y| = m_2$ and $|x| = m_1$

- 5. Player 1 "wins" by picking *k*, which may be a function of *n,x,y*, and *z* such that xy^kz is not in L.

xy consists entirely of first set of a's

$xy^0z \in L \Rightarrow a^{n-m_2} b^n a^n b^n \notin L$

$m_2 > 0$

Power of abstraction & Combining theorems

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Exercise:

- $L = \{ 0^{2i}1^{i}2^{i} \mid i>0 \}$

More examples…

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